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Noncommutative supergravity in D=3 and D=4

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Abstract: We present a noncommutative $D = 3$, $N = 1$ supergravity, invariant under diffeomorphisms, local $U(1,1)$ noncommutative $\star$-gauge transformations and local $\star$-supersymmetry. Its commutative limit is the usual $D = 3$ pure supergravity, without extra fields.

A noncommutative deformation of $D = 4$, $N = 1$ supergravity is also obtained, reducing to the usual simple supergravity in the commutative limit. Its action is invariant under diffeomorphisms and local $GL(2,C)$ $\star$-gauge symmetry. The supersymmetry of the commutative action is broken by noncommutativity. Local $\star$-supersymmetry invariance can be implemented in a noncommutative $D = 4$, $N = 1$ supergravity with chiral gravitino and complex vierbein.

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Contents

1 Introduction

Gravity theories on $D = 4$ twisted spaces have been constructed in the past in the context of particular quantum groups [1] and more recently in the twisted noncommutative geometry setting [2–4]. In this setting the deformed theory is invariant under $\star$-diffeomorphisms, but in [3] no gauge invariance on the tangent space (generalizing local Lorentz symmetry) is incorporated, and therefore coupling to fermions could not be implemented. A local symmetry, enlarging the local $SO(3,1)$ symmetry of $D = 4$ Einstein gravity to $GL(2,C)$, has been considered in the approach of Chamseddine [2]. The resulting theory has a complicated classical limit, with two vielbeins (or, equivalently, a complex vielbein). Noncommutative gravities in lower dimensions have been studied in [5] (D=2) and in [6, 7] (D=3).

In [8] we have proposed a noncommutative gravity, coupled to fermions, and reducing in the commutative limit to ordinary gravity + fermions, without extra fields (in particular without an extra graviton). This is achieved by imposing a noncommutative charge conjugation condition on the bosonic fields, consistent with the $\star$-gauge transformations. One can also impose a noncommutative generalization of the Majorana condition on the fermions, compatible with the $\star$-gauge transformations.

In this paper we present the noncommutative extensions of locally supersymmetric $D = 3$ and $D = 4$ gravity theories. The noncommutativity is given by a $\star$-product associated to a very general class of twists. This $\star$-product can also be x-dependent. The deformed supergravity actions are constructed with a cyclic integral. As a particular case we obtain noncommutative supergravities where noncommutativity is realized with the Moyal-Groenewald $\star$-product.

For $D = 3$ the situation is easier, since in three dimensions gravity becomes essentially a Chern-Simons gauge theory. The noncommutative extension of a particular $AdS(3)$ supergravity in three dimensions has been studied in [9].

Here we discuss $D = 3$, $N = 1$ supergravity without cosmological term. The noncommutative geometric action is constructed directly by generalizing the usual $D = 3$ supergravity action, without reference to the Chern-Simons action. The noncommutative theory is invariant under diffeomorphisms, local $U(1,1)$ $\star$-gauge symmetry and $\star$-supersymmetry.

We then propose an action for a noncommutative deformation of $D = 4$, $N = 1$ supergravity, invariant under diffeomorphisms and local $GL(2,C)$ $\star$-gauge transformations, but without $\star$-supersymmetry. In this case noncommutativity breaks the local supersymmetry of the commutative theory. The commutative $\theta \to 0$ limit is the usual $D = 4$, $N = 1$ simple supergravity, with a Majorana gravitino.

We can obtain local $\star$-supersymmetry invariance of the noncommutative action if we impose a Weyl condition on the fermions, rather than a Majorana condition. This leads to a noncommutative supergravity whose $\theta \to 0$ limit is a chiral $D = 4$, $N = 1$ supergravity with two vierbein fields (or a complex vierbein) and a left-handed gravitino.

The paper is organized as follows. In section 2 we discuss three dimensional noncommutative simple supergravity, in first order formalism. In section 3 we present the index-free formulation of usual $D = 4$, $N = 1$ supergravity, exploiting the Clifford algebra representation of boson fields, thus preparing the ground for its noncommutative extension.

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In this setting the supersymmetry of the action becomes quite easy to prove. In section 4 we consider noncommutative first order $D = 4$, $N = 1$ supergravity, and prove its local $\star$-invariances. Section 5 contains some conclusions. In appendix A we collect a few useful results of twist differential geometry. Conventions, $D = 3$ and $D = 4$ gamma matrices properties are summarized in appendices B and C.

2 Noncommutative $D = 3$, $N = 1$ supergravity

2.1 Action

Using the $*$-exterior product of twist differential geometry (see appendix A), we extend the usual action of $D = 3$, $N = 1$ supergravity to its noncommutative version. In index-free notation:

$$S = -2 \int Tr \left[R(\Omega) \wedge_\star V + i\rho \wedge_\star \bar{\psi} \right] \tag{2.1}$$

The fundamental fields are the 1-forms Ω (spin connection), V (vielbein) and gravitino ψ. The curvature 2-form R and the gravitino curvature ρ are defined by

$$R = d\Omega - \Omega \wedge_\star \Omega, \qquad \rho \equiv D\psi = d\psi - \Omega \wedge_\star \psi \tag{2.2}$$

with

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega 1, \qquad V = V^a \gamma_a + iv1 \tag{2.3}$$

and thus are 2×2 matrices with spinor indices, see appendix B for $D = 3$ gamma matrix conventions and useful relations. The Dirac conjugate is defined as usual: $\bar{\psi} = \psi^\dagger \gamma_0$. Then $(D\psi) \wedge_\star \bar{\psi}$ is also a matrix in the spinor representation, and the trace Tr is taken on this representation. Using the $D = 3$ gamma matrix identity:

$$Tr(\gamma_a \gamma_b \gamma_c) = -2\varepsilon_{abc} \tag{2.4}$$

allows to rewrite the action in terms of component fields:

$$S = \int R^{ab} \wedge_\star V^c \varepsilon_{abc} + 4r \wedge_\star v + 2i\bar{\psi} \wedge_\star \rho \tag{2.5}$$

with

$$R \equiv \frac{1}{4} R^{ab} \gamma_{ab} + ir1, \tag{2.6}$$

and

$$R^{ab} = d\omega^{ab} - \frac{1}{2}\omega^a{}_c \wedge_\star \omega^{cb} + \frac{1}{2}\omega^b{}_c \wedge_\star \omega^{ca} - i(\omega^{ab} \wedge_\star \omega + \omega \wedge_\star \omega^{ab}), \tag{2.7}$$

$$r = d\omega - i\omega \wedge_\star \omega - \frac{i}{8}\omega^{ab} \wedge_\star \omega_{ab} \tag{2.8}$$

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2.2 Hermiticity conditions and reality of the action

Hermiticity conditions can be imposed on V and Ω:

$$\gamma_0 V \gamma_0 = V^\dagger, \quad \gamma_0 \Omega \gamma_0 = \Omega^\dagger \tag{2.9}$$

Moreover it is easy to verify that:

$$\gamma_0 R \gamma_0 = R^\dagger, \quad \gamma_0 [\rho \wedge_\star \bar{\psi}] \gamma_0 = [\psi \wedge_\star \bar{\rho}]^\dagger \tag{2.10}$$

with

$$\bar{\rho} = d\bar{\psi} - \bar{\psi} \wedge_\star \Omega \tag{2.11}$$

Note also that up to boundary terms

$$\int Tr[\rho \wedge_\star \bar{\psi}] = \int Tr[\psi \wedge_\star \bar{\rho}] = -\int \bar{\psi} \wedge_\star \rho = -\int \bar{\rho} \wedge_\star \psi \tag{2.12}$$

where we have used the cyclicity of Tr and the graded cyclicity of the integral. For example the first equality holds because

$$\int Tr[\rho \wedge_\star \bar{\psi}] = \int Tr[d(\psi \wedge_\star \bar{\psi}) + \psi \wedge_\star \bar{\rho}] \tag{2.13}$$

These formulae can be used to check that the action (2.1) is real.

The hermiticity conditions (2.9) imply that the component fields V^a, v, ω^{ab}, ω are real.

2.3 Field equations

Using the cyclicity of Tr and the graded cyclicity of the integral in (2.1), the variation of V , Ω and $\bar{\psi}$ yield respectively the noncommutative Einstein equation, torsion equation and gravitino equation in index-free form:

$$R = 0 \tag{2.14}$$

$$dV - \Omega \wedge_\star V - V \wedge_\star \Omega - i\psi \wedge_\star \bar{\psi} = 0 \tag{2.15}$$

$$\rho = 0 \tag{2.16}$$

The noncommutative torsion two-form is defined by:

$$T \equiv T^a \gamma_a + it1 \equiv dV - \Omega \wedge_\star V - V \wedge_\star \Omega \tag{2.17}$$

or, in component fields:

$$\begin{aligned} T^a &= dV^a - \frac{1}{2}\left(\omega^a{}_b \wedge_\star V^b - V^b \wedge_\star \omega^a{}_b\right) + \frac{i}{4}\epsilon^{abc}\left(\omega_{bc} \wedge_\star v + v \wedge_\star \omega_{bc}\right) \\ &\quad - i\left(\omega \wedge_\star V^a + V^a \wedge_\star \omega\right) \end{aligned} \tag{2.18}$$

$$t = dv - \frac{i}{4}\epsilon_{abc}\left(\omega^{ab} \wedge_\star V^c + V^c \wedge_\star \omega^{ab}\right) - i\omega \wedge_\star v - iv \wedge_\star \omega \tag{2.19}$$

The torsion equation $T = i\psi \wedge_\star \bar{\psi}$ (2.15) yields:

$$T^a = \frac{i}{2} Tr(\psi \wedge_\star \bar{\psi}\gamma^a), \quad t = \frac{1}{2} Tr(\psi \wedge_\star \bar{\psi}) \tag{2.20}$$

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2.4 Bianchi identities

From their definition, the curvatures R, ρ and the torsion T satisfy the identities

$$dR = -R \wedge_\star \Omega + \Omega \wedge_\star R \tag{2.21}$$

$$d\rho = -R \wedge_\star \psi + \Omega \wedge_\star \rho \tag{2.22}$$

$$dT = \Omega \wedge_\star T - T \wedge_\star \Omega - R \wedge_\star V + V \wedge_\star R \tag{2.23}$$

The terms on right-hand sides with the spin connection Ω reconstruct covariant derivatives on curvatures and torsion, so that the identities take the form

$$DR = 0, \quad D\rho = -R \wedge_\star \psi, \quad DT = -R \wedge_\star V + V \wedge_\star R \tag{2.24}$$

2.5 Invariances

The action (2.1) is invariant under:

i) **Diffeomorphisms**: generated by the usual Lie derivative. Indeed the action is the integral of a 3-form on a 3-manifold,[1]

$$\int \mathcal{L}_v(\text{3-form}) = \int (i_v d + d i_v)(\text{3-form}) = \int d(i_v(\text{3-form})) = \text{boundary term} \tag{2.25}$$

since d(3-form) $= 0$ on a 3-dimensional manifold. We have constructed a geometric lagrangian where the fields are exterior forms and the $\star$-product is given by the Lie derivative action of the twist on forms. The twist $\mathcal{F}$ in general is not invariant under the diffeomorphism $\mathcal{L}_v$. However we can consider the $\star$-diffeomorphisms of ref. [3] (see also [14], section 8.2.4), generated by the $\star$-Lie derivative. This latter acts trivially on the twist $\mathcal{F}$ but satisfies a deformed Leibniz rule. $\star$-Lie derivatives generate infinitesimal noncommutative diffeomorphisms and leave invariant the action and the twist. They are noncommutative symmetries of our action.

Finally in our geometric action no coordinate indices μ, ν appear, and this implies invariance of the action under (undeformed) general coordinate transformations.[2] Otherwise stated every contravariant tensor index $^\mu$ is contracted with the corresponding covariant tensor index $_\mu$, for example $X_a = X_a^\mu \partial_\mu$ and $V^a = V_\mu^a dx^\mu$.

ii) **Local** $SO(1,2) \times U(1) \approx U(1,1)$ variations:

$$\delta_\epsilon V = -V \star \epsilon + \epsilon \star V, \quad \delta_\epsilon \Omega = d\epsilon - \Omega \star \epsilon + \epsilon \star \Omega, \quad \delta_\epsilon \psi = \epsilon \star \psi, \quad \delta_\epsilon \bar{\psi} = -\bar{\psi} \star \epsilon \tag{2.26}$$

[1] In order to show that the integrand is a globally defined 3-form we need to assume that the vielbein one-form V^a is globally defined (and therefore that the manifold is parallelizable), the twisted exterior product being globally defined (because the twist is globally defined). If this is the case, then due to the local $SO(1,2) \times U(1)$ invariance (see point ii) below) the action is independent of the vielbein used. On the other hand, if the vielbein V^a is only locally defined in open coverings of the manifold, then we cannot construct a global 3-form, since the local $SO(1,2) \times U(1)$ invariance holds only under integration.

[2] General coordinate transformations are diffeomorphisms of an open coordinate neighbourhood of the manifold, not of the whole manifold.

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with

$$\epsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon 1 \tag{2.27}$$

satisfying the hermiticity condition:

$$\gamma_0 \epsilon \gamma_0 = \epsilon^\dagger \tag{2.28}$$

This condition implies reality of the component gauge parameters ε^{ab}, ε.

The invariance of (2.1) can be easily checked noting that

$$\delta_\epsilon R = -R \star \epsilon + \epsilon \star R, \quad \delta_\epsilon \rho = \epsilon \star \rho, \quad \delta_\epsilon(\rho \wedge_\star \bar{\psi}) = -\rho \wedge_\star \bar{\psi} \star \epsilon + \epsilon \star \rho \wedge_\star \bar{\psi} \tag{2.29}$$

and using the cyclicity of the trace Tr and the graded cyclicity of the integral.

iii) **Local N=1 $\star$-supersymmetry** variations:

$$\delta_\epsilon V = i(\epsilon \star \bar{\psi} - \psi \star \bar{\epsilon}), \quad \delta_\epsilon \psi = d\epsilon - \Omega \star \epsilon \tag{2.30}$$

where now ϵ is a spinorial parameter. Notice that Ω is not varied: we are working in 1.5 - order formalism, i.e. we are considering Ω as already satisfying its own equation of motion (2.15). Then the variation of the action due to the supersymmetry variation of Ω vanishes, since it is proportional to the Ω field equation. The variations (2.30) imply:

$$\delta_\epsilon \bar{\psi} = d\bar{\epsilon} + \bar{\epsilon} \star \Omega, \quad \delta_\epsilon \rho = -R \star \epsilon, \quad \delta_\epsilon \bar{\rho} = \bar{\epsilon} \star R \tag{2.31}$$

The action varies as:

$$\delta_\epsilon S = -2i \int Tr[R \wedge_\star (-\psi \star \bar{\epsilon} + \epsilon \star \bar{\psi}) + (-R \star \epsilon) \wedge_\star \bar{\psi} + \rho \wedge_\star (d\bar{\epsilon} + \bar{\epsilon} \star \Omega)] \tag{2.32}$$

After integrating by parts the term with $d\bar{\epsilon}$, using the Bianchi identity for $d\rho$ (2.22) and reordering the $\rho\bar{\epsilon}\Omega$ term using the cyclicity of Tr and graded cyclicity of the integral, all terms are seen to cancel. Thus the action (where Ω is resolved via its equation of motion, i.e. in second order formalism) is invariant under the local $\star$-supersymmetry transformations (2.30), up to boundary terms.

On the component fields, the $U(1,1)$ transformation rules are:

$$\begin{aligned}
\delta_\epsilon V^a &= \frac{1}{2}\varepsilon^a{}_b \star V^b + \frac{1}{2} V^b \star \varepsilon^a{}_b + \frac{i}{4}\epsilon^{abc}\left(v \star \varepsilon_{bc} - \varepsilon_{bc} \star v\right) + i\left(\varepsilon \star V^a - V^a \star \varepsilon\right) \\
\delta_\epsilon v &= -\frac{i}{4}\epsilon_{abc}\left(V^a \star \varepsilon^{bc} - \varepsilon^{bc} \star V^a\right) - i\left(v \star \varepsilon - \varepsilon \star v\right) \\
\delta_\epsilon \omega^{ab} &= d\varepsilon^{ab} + \omega^{c[a} \star \varepsilon_c{}^{b]} - \varepsilon^{c[a} \star \omega_c{}^{b]} - i\left(\omega^{ab} \star \varepsilon - \varepsilon \star \omega^{ab}\right) - i\left(\omega \star \varepsilon^{ab} - \varepsilon^{ab} \star \omega\right) \\
\delta_\epsilon \omega &= -d\epsilon - \frac{i}{8}\left(\omega^{ab} \star \varepsilon_{ab} - \varepsilon^{ab} \star \omega_{ab}\right) - i\left(\omega \star \varepsilon - \varepsilon \star \omega\right) \\
\delta_\epsilon \psi &= \frac{1}{4}\varepsilon^{ab}\gamma_{ab} \star \psi + i\varepsilon \star \psi
\end{aligned} \tag{2.33}$$

and the supersymmetry variations are:

$$\begin{aligned}\delta_\epsilon V^a &= \frac{i}{2}Tr(\epsilon \star \bar{\psi}\gamma^a - \psi \star \bar{\epsilon}\gamma^a)\\ \delta_\epsilon v &= \frac{1}{2}Tr(\epsilon \star \bar{\psi} - \psi \star \bar{\epsilon})\\ \delta_\epsilon \psi &= d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab} \star \epsilon - i\omega \star \epsilon \end{aligned} \tag{2.34}$$

Finally, it is a straightforward exercise to check that the hermiticity conditions on the fields and on the parameters are consistent with the $\star$-gauge and $\star$-supersymmetry variations.

2.6 Commutative limit $\theta \to 0$

In the commutative limit the action (2.5) reduces to

$$S_{\theta=0} = \int R^{ab} \wedge V^c \varepsilon_{abc} + 4r \wedge v + 2i\bar{\psi} \wedge \rho \tag{2.35}$$

with

$$R^{ab} = d\omega^{ab} - \omega^a_{\ c} \wedge \omega^{cb}, \quad r = d\omega \tag{2.36}$$

$$\rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab} \wedge \psi - i\omega \wedge \psi \tag{2.37}$$

The $\theta = 0$ field equations imply, as in the noncommutative case, that all curvatures R^{ab}, r, ρ vanish. The $\theta = 0$ torsion constraints become:

$$dV^a - \omega^a_{\ b} \wedge V^b = \frac{i}{2}\bar{\psi}\gamma^a \wedge \psi, \quad dv = \frac{1}{2}\bar{\psi} \wedge \psi \tag{2.38}$$

The term $r \wedge v = d\omega \wedge v$ in the action (2.35) can be integrated by parts. Using now the second torsion constraint dv can be substituted by $(1/2)(\bar{\psi} \wedge \psi)$, and the whole term exactly cancels the $\bar{\psi}\omega\psi$ term coming from the third term in (2.35). Thus the $\theta = 0$ action becomes

$$S_{\theta=0} = \int R^{ab} \wedge V^c \varepsilon_{abc} + 2i\bar{\psi} \wedge \left(d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab} \wedge \psi\right) \tag{2.39}$$

and does not contain any more the fields ω and v. In fact it coincides with the usual $D = 3$ pure supergravity action, involving only the dreibein V^a and the gravitino ψ. One can at this point use also the first torsion constraint to express ω^{ab} in terms of the dreibein, retrieving the classical action in second order formalism.

Note: the second torsion constraint in (2.38) implies that $\bar{\psi} \wedge \psi$ must be closed, which is true on-shell since $d(\bar{\psi} \wedge \psi) = \bar{\rho} \wedge \psi - \bar{\psi} \wedge \rho$.

3 Classical $D = 4$, $N = 1$ supergravity

The $D = 4$, $N = 1$ simple supergravity action can be written in index-free notation as follows:

$$S = \int Tr \left[iR(\Omega) \wedge V \wedge V\gamma_5 - 2(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}) \wedge V\gamma_5\right] \tag{3.1}$$

The fundamental fields are the 1-forms Ω (spin connection), V (vielbein) and gravitino ψ. The curvature 2-form R and the gravitino curvature ρ are defined by

$$R = d\Omega - \Omega \wedge \Omega, \qquad \rho \equiv D\psi = d\psi - \Omega\psi, \quad \bar{\rho} = D\bar{\psi} = d\bar{\psi} - \bar{\psi} \wedge \Omega \tag{3.2}$$

with

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab}, \qquad V = V^a\gamma_a \tag{3.3}$$

and thus are 4×4 matrices with spinor indices. See appendix C for $D = 4$ gamma matrix conventions and useful relations. The Dirac conjugate is defined as usual: $\bar{\psi} = \psi^\dagger\gamma_0$. Then also $\rho \wedge \bar{\psi}$ and $\psi \wedge \bar{\rho}$ are matrices in the spinor representation, and the trace Tr is taken on this representation. The gravitino field satisfies the Majorana condition:

$$\psi^\dagger\gamma_0 = \psi^T C \tag{3.4}$$

where C is the $D = 4$ charge conjugation matrix, antisymmetric and squaring to -1.

Using the $D = 4$ gamma matrix trace identity:

$$Tr(\gamma_{ab}\gamma_c\gamma_d\gamma_5) = -4i\varepsilon_{abcd} \tag{3.5}$$

leads to the usual supergravity action in terms of the component fields V^a, ω^{ab}:

$$S = \int R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} - 4\bar{\psi} \wedge \gamma_5\gamma_a\rho \wedge V^a \tag{3.6}$$

with

$$R \equiv \frac{1}{4}R^{ab}\gamma_{ab}, \quad R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \tag{3.7}$$

We have also used

$$\bar{\rho}\gamma_5\gamma_a\psi = \bar{\psi}\gamma_5\gamma_a\rho \tag{3.8}$$

due to ψ and ρ being Majorana spinors.[3]

3.1 Field equations and Bianchi identities

Using the cyclicity of the Tr in the action (3.1), the variation on V, Ω and ψ yield respectively the Einstein equation, the torsion equation and the gravitino equation in index-free form:

$$\begin{aligned} Tr\left[\gamma_a\gamma_5\left(-iV \wedge R - iR \wedge V + 2\left(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}\right)\right)\right] &= 0 && (3.9)\\ Tr\left[\gamma_{ab}\gamma_5\left(iT \wedge V - iV \wedge T + 2\psi \wedge \bar{\psi} \wedge V - 2V \wedge \psi \wedge \bar{\psi}\right)\right] &= 0 && (3.10)\\ V \wedge D\psi &= 0 && (3.11) \end{aligned}$$

where the torsion $T = T^a\gamma_a$ is defined as:

$$T \equiv dV - \Omega \wedge V - V \wedge \Omega \tag{3.12}$$

[3]Then the two addends in the fermionic part of the action (3.1) are equal, so that we could have used only one of them, with factor -4. However in the noncommutative extension both will be necessary.

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The solution of the torsion equation (3.10) is given by:

$$T = i[\psi \wedge \bar{\psi}, \gamma_5]\gamma_5 = i\psi \wedge \bar{\psi} - i\gamma_5\psi \wedge \bar{\psi}\gamma_5 \tag{3.13}$$

Upon use of the Fierz identity for Majorana spinor one-forms:

$$\psi \wedge \bar{\psi} = \frac{1}{4}\gamma_a\bar{\psi}\gamma^a \wedge \psi - \frac{1}{8}\gamma_{ab}\bar{\psi}\gamma^{ab} \wedge \psi \tag{3.14}$$

the torsion is seen to satisfy the familiar condition

$$T \equiv T^a\gamma_a = \frac{i}{2}\bar{\psi}\gamma^a \wedge \psi\gamma_a \tag{3.15}$$

Finally, the Bianchi identities for the curvatures and the torsion are:

$$dR = -R \wedge \Omega + \Omega \wedge R \tag{3.16}$$
$$d\rho = -R \wedge \psi + \Omega \wedge \rho, \quad d\bar{\rho} = \bar{\psi} \wedge R - \bar{\rho} \wedge \Omega \tag{3.17}$$
$$dT = -R \wedge V + \Omega \wedge T - T \wedge \Omega + V \wedge R \tag{3.18}$$

The terms with the spin connection Ω reconstruct covariant derivatives of the curvatures and the torsion.

3.2 Invariances

We know that the classical supergravity action (3.6) is invariant under general coordinate transformations, under local Lorentz rotations and under local supersymmetry transformations. It is of interest to write the transformation rules of the fields in the index-free notation, so as to verify the invariances directly on the index-free action (3.1).

Local Lorentz rotations

$$\delta_\epsilon V = -[V, \epsilon], \quad \delta_\epsilon\Omega = d\epsilon - [\Omega, \epsilon], \quad \delta_\epsilon\psi = \epsilon\psi, \quad \delta_\epsilon\bar{\psi} = -\bar{\psi}\epsilon \tag{3.19}$$

with

$$\epsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} \tag{3.20}$$

The invariance can be directly checked on the action (3.1) noting that

$$\delta_\epsilon R = -[R, \epsilon], \quad \delta_\epsilon D\psi = \epsilon D\psi, \quad \delta_\epsilon D\bar{\psi} = -(D\bar{\psi})\epsilon \tag{3.21}$$

using the cyclicity of the trace Tr (on spinor indices) and the fact that ϵ commutes with γ_5. The Lorentz rotations close on the Lie algebra:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{[\epsilon_2, \epsilon_1]} \tag{3.22}$$

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Local supersymmetry. The supersymmetry variations are:

$$\delta_\epsilon V = i[\epsilon\bar{\psi} - \psi\bar{\epsilon}, \gamma_5]\gamma_5, \quad \delta_\epsilon \psi = D\epsilon \equiv d\epsilon - \Omega\epsilon \tag{3.23}$$

where now ϵ is a spinorial parameter (satisfying the Majorana condition). Notice that again Ω is not varied since we work in 1.5 - order formalism, i.e. Ω satisfies its own equation of motion (3.10).

The commutator of $\epsilon\bar{\psi} - \psi\bar{\epsilon}$ with γ_5 in the supersymmetry variation of V eliminates the terms even in γ_a in the Fierz expansion of two generic anticommuting spinors (see appendix C). Moreover, since ϵ and ψ are Majorana spinors, the combination $\epsilon\bar{\psi} - \psi\bar{\epsilon}$ ensures that only the γ_a component survives. Then (3.23) reproduce the usual supersymmetry variations (see below).

The variations (3.23) imply:

$$\delta_\epsilon \bar{\psi} = D\bar{\epsilon} \equiv d\bar{\epsilon} + \bar{\epsilon}\Omega, \quad \delta_\epsilon \rho = -R\epsilon, \quad \delta_\epsilon \bar{\rho} = \bar{\epsilon}R \tag{3.24}$$

Then the action varies as:

$$\begin{aligned}\delta_\epsilon S = \int & 2\, Tr[R \wedge (\psi\bar{\epsilon} - \epsilon\bar{\psi}) \wedge V\gamma_5 + R \wedge V \wedge (\psi\bar{\epsilon} - \epsilon\bar{\psi})\gamma_5] - \\ & -2\, Tr\left[\left(-R\epsilon \wedge \bar{\psi} \wedge V + \rho \wedge (d\bar{\epsilon} + \bar{\epsilon}\Omega) \wedge V + (d\epsilon - \Omega\epsilon) \wedge \bar{\rho} \wedge V + \psi \wedge \bar{\epsilon}R \wedge V\right)\gamma_5\right] \\ & +2i\, Tr\left[\left(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}\right)\left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right)\gamma_5 - \left(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}\right)\gamma_5\left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right)\right]\end{aligned} \tag{3.25}$$

After integrating by parts the terms with $d\epsilon$ and $d\bar{\epsilon}$, and using the Bianchi identity (3.17) for $d\rho$ the variation becomes:

$$\begin{aligned}\delta_\epsilon S = \int & 2\, Tr\left[R \wedge \left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right) \wedge V\gamma_5 + R \wedge V \wedge \left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right)\gamma_5\right] - \\ & -2\, Tr\Big[\Big(- R\epsilon \wedge \bar{\psi} \wedge V + \rho \wedge \bar{\epsilon}\Omega \wedge V - \Omega\epsilon \wedge \bar{\rho} \wedge V + \psi \wedge \bar{\epsilon}R \wedge V + \\ & \qquad + (R \wedge \psi - \Omega \wedge \rho)\,\bar{\epsilon} \wedge V - \rho\bar{\epsilon} \wedge (T + \Omega \wedge V + V \wedge \Omega) - \\ & \qquad - \epsilon\left(-\bar{\rho} \wedge \Omega + \bar{\psi} \wedge \rho\right) \wedge V - \epsilon\bar{\rho} \wedge (T + \Omega \wedge V + V \wedge \Omega)\Big)\gamma_5\Big] + \\ & +2i\, Tr\left[\left(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}\right)\left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right)\gamma_5 - \left(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}\right)\gamma_5\left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right)\right]\end{aligned} \tag{3.26}$$

where we have substituted dV by $T + \Omega \wedge V + V \wedge \Omega$ (torsion definition). Using now the cyclicity of Tr , and the fact that γ_5 anticommutes with V and commutes with Ω, all terms can be easily checked to cancel, except those containing the torsion T and the last line (four-fermion terms).

Once we make use of the torsion equation ((3.13) to express T in terms of gravitino fields, the variation reduces to:

$$\begin{aligned}\delta_\epsilon S = 2i \int & Tr\Big[\rho\bar{\epsilon} \wedge \left(\psi \wedge \bar{\psi}\gamma_5 - \gamma_5\psi \wedge \bar{\psi}\right) + \epsilon\bar{\rho} \wedge \left(\psi \wedge \bar{\psi}\gamma_5 - \gamma_5\psi \wedge \bar{\psi}\right) \\ & + \left(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}\right) \wedge \left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right)\gamma_5 - \left(\rho \wedge \bar{\psi} + \psi \wedge \bar{\rho}\right) \wedge \gamma_5\left(\psi\bar{\epsilon} - \epsilon\bar{\psi}\right)\Big]\end{aligned} \tag{3.27}$$

Finally, carrying out the trace on spinor indices results in

$$\begin{aligned}\delta_\epsilon S = 2i \int & (\bar{\psi}\epsilon - \bar{\epsilon}\psi) \wedge (\bar{\psi}\gamma_5 \wedge \rho - \bar{\rho}\gamma_5 \wedge \psi) + (\bar{\psi} \wedge \rho - \bar{\rho} \wedge \psi) \wedge (\bar{\psi}\gamma_5\epsilon - \bar{\epsilon}\gamma_5\psi) \\ & +(\bar{\epsilon}\rho - \bar{\rho}\epsilon) \wedge (\bar{\psi}\gamma_5 \wedge \psi) + (\bar{\rho}\gamma_5\epsilon - \bar{\epsilon}\gamma_5\rho) \wedge (\bar{\psi} \wedge \psi)\end{aligned} \tag{3.28}$$

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Each factor between parentheses vanishes, due to all spinors being Majorana spinors. This proves the invariance of the classical supergravity action under the local supersymmetry variations (3.23).

On the component fields, the Lorentz transformations (3.19) read:

$$\begin{aligned}\delta_\epsilon V^a &= \varepsilon^a{}_b V^b \\ \delta_\epsilon \omega^{ab} &= d\varepsilon^{ab} + \varepsilon^{ac}\omega_c{}^b - \varepsilon^{bc}\omega_c{}^a \\ \delta_\epsilon \psi &= \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi\end{aligned} \tag{3.29}$$

and the supersymmetry variations (3.23) become:

$$\begin{aligned}\delta_\epsilon V^a &= i\bar{\epsilon}\gamma^a\psi \\ \delta_\epsilon \psi &= d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon\end{aligned} \tag{3.30}$$

4 Noncommutative $D = 4$, $N = 1$ supergravity

4.1 Action and $GL(2,C)$ $\star$-gauge symmetry

A noncommutative generalization of the $D = 4$, $N = 1$ simple supergravity action is obtained by replacing exterior products by $\star$-exterior products in (3.1):

$$S = \int Tr\left[iR(\Omega) \wedge_\star V \wedge_\star V\gamma_5 + 2(\rho \wedge_\star \bar{\psi} + \psi \wedge_\star \bar{\rho}) \wedge_\star V\gamma_5\right] \tag{4.1}$$

where the curvature 2-form R and the gravitino curvature ρ are defined as:

$$R = d\Omega - \Omega \wedge_\star \Omega, \qquad \rho \equiv D\psi = d\psi - \Omega \star \psi \tag{4.2}$$

Almost all formulae of the commutative case continue to hold, with ordinary products replaced by $\star$-products and $\star$-exterior products. However, the expansion of the fundamental fields on the Dirac basis of gamma matrices must now include new contributions; more precisely the spin connection contains all even gamma matrices and the vielbein contains all odd gamma matrices:

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega 1 + \tilde{\omega}\gamma_5, \qquad V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5 \tag{4.3}$$

The one-forms Ω and V are thus also 4×4 matrices with spinor indices. Similarly for the curvature:

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + ir1 + \tilde{r}\gamma_5 \tag{4.4}$$

and for the gauge parameter:

$$\epsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon 1 + \tilde{\varepsilon}\gamma_5 \tag{4.5}$$

Indeed now the $\star$-gauge variations read:

$$\delta_\epsilon V = -V \star \epsilon + \epsilon \star V, \quad \delta_\epsilon\Omega = d\epsilon - \Omega \star \epsilon + \epsilon \star \Omega, \quad \delta_\epsilon\psi = \epsilon \star \psi, \quad \delta_\epsilon\bar{\psi} = -\bar{\psi} \star \epsilon \tag{4.6}$$

and in the variations for V and Ω also anticommutators of gamma matrices appear, due to the noncommutativity of the $\star$-product. Since for example the anticommutator $\{\gamma_{ab}, \gamma_{cd}\}$ contains 1 and γ_5, we see that the corresponding fields must be included in the expansion of Ω. Similarly, V must contain a $\gamma_a\gamma_5$ term due to $\{\gamma_{ab}, \gamma_c\}$. Finally, the composition law for gauge parameters becomes:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_2 \star \epsilon_1 - \epsilon_1 \star \epsilon_2} \tag{4.7}$$

so that ϵ must contain the 1 and γ_5 terms, since they appear in the composite parameter $\epsilon_2 \star \epsilon_1 - \epsilon_1 \star \epsilon_2$.

The invariance of the noncommutative action (4.1) under the $\star$-gauge variations is demonstrated in exactly the same way as for the commutative case, noting that

$$\delta_\epsilon R = -R \star \epsilon + \epsilon \star R, \quad \delta_\epsilon D\psi = \epsilon \star D\psi, \quad \delta_\epsilon((D\psi) \wedge_\star \bar{\psi}) = -(D\psi) \wedge_\star \bar{\psi} \star \epsilon + \epsilon \star (D\psi) \wedge_\star \bar{\psi} \tag{4.8}$$

and using now, besides the cyclicity of the trace Tr and the fact that ϵ still commutes with γ_5, also the graded cyclicity of the integral.

4.2 Local $\star$-supersymmetry

The $\star$-supersymmetry variations are obtained from the classical ones using $\star$-products:

$$\delta_\epsilon V = i[\epsilon \star \bar{\psi} - \psi \star \bar{\epsilon}, \gamma_5]\gamma_5 \quad \delta_\epsilon \psi = d\epsilon - \Omega \star \epsilon \tag{4.9}$$

where ϵ is a spinorial parameter. Under these variations the noncommutative action varies as given in (3.28), with ordinary products substituted with $\star$-products. Indeed the algebra is identical, since γ_5 still anticommutes with V and commutes with Ω, and we can use the cyclicity of Tr and graded cyclicity of the integral.

The question is now: does this variation vanish? Classically it vanishes because of the Majorana condition on the spinors (gravitino and supersymmetry gauge parameter). We recall the noncommutative generalization of the Majorana condition, consistent with the $*$-gauge transformations [8]:

$$\psi^c_\theta = \psi_{-\theta}, \qquad \psi^c \equiv C(\bar{\psi})^T \tag{4.10}$$

This condition involves the θ dependence of the fields,[4] and is consistent with the $\star$-gauge transformations only if the gauge parameter satisfies the charge conjugation condition [8]:

$$C\epsilon_\theta C = \epsilon^T_{-\theta} \tag{4.11}$$

The NC Majorana condition (4.10) is consistent also with $\star$-supersymmetry transformations if the supersymmetry parameter is Majorana, and the bosonic fields satisfy the charge conjugation conditions

$$C\Omega_\theta C = \Omega^T_{-\theta}, \quad CV_\theta C = V^T_{-\theta} \tag{4.12}$$

[4] The fields can be formally expanded in powers of θ: in principle this picture would introduce infinitely many fields, one for each power of θ. However the Seiberg-Witten map [10, 11] can be used to express all fields in terms of the classical one, ending up with a finite number of fields.

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Now consider the first term in the supersymmetry variation of the action (for the other three terms the reasoning is identical):

$$2i \int (\bar{\psi} \star \epsilon - \bar{\epsilon} \star \psi) \wedge_\star (\bar{\psi}\gamma_5 \wedge_\star \rho - \bar{\rho}\gamma_5 \wedge_\star \psi) \tag{4.13}$$

If ψ and ϵ are noncommutative Majorana fermions, they satisfy the relations:

$$\bar{\psi} \star \epsilon = \bar{\epsilon}_{-\theta} \star_{-\theta} \psi_{-\theta}, \quad \bar{\psi}\gamma_5 \wedge_\star \rho = \bar{\rho}_{-\theta}\gamma_5 \wedge_{-\theta} \psi_{-\theta} \tag{4.14}$$

and one sees that (4.13) does not vanish anymore (although it vanishes in the commutative limit). Thus the NC Majorana condition does not ensure the local $\star$-supersymmetry invariance of the action in (4.1). In fact, the local supersymmetry of the commutative action is broken by noncommutativity.

There is another condition that we can impose on fermi fields, the Weyl condition, still consistent with the $\star$-symmetry structure of the action:

$$\gamma_5\psi = \psi, \quad \gamma_5\epsilon = \epsilon \tag{4.15}$$

i.e. all fermions are left-handed (so that their Dirac conjugates $\bar{\psi}$ and $\bar{\epsilon}$ are right-handed). In this case the local $\star$-supersymmetry variation vanishes because in all the fermion bilinears the γ_5 matrices can be omitted, and the product of a right-handed spinor with a left-handed spinor vanishes. Thus the noncommutative supergravity action (4.1) with Weyl fermions is locally supersymmetric.

Note that now we cannot impose the charge conjugation relations (4.12) on the bosonic fields : indeed $\star$-supersymmetry links together these relations with the NC Majorana condition, which is not compatible in $D = 4$ with the Weyl condition (as in the classical case).

The $\theta \to 0$ limit of this chiral noncommutative theory is a complex version of the so-called $D = 4$, $N = 1$ Weyl supergravity and is discussed in section 4.6 below.

4.3 Hermiticity conditions and reality of the action

Hermiticity conditions can be imposed on V, Ω and the gauge parameter ϵ:

$$\gamma_0 V \gamma_0 = V^\dagger, \quad -\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \quad -\gamma_0 \epsilon \gamma_0 = \epsilon^\dagger \tag{4.16}$$

Moreover it is easy to verify that:

$$\gamma_0 [\rho \wedge_\star \bar{\psi}] \gamma_0 = [\psi \wedge_\star \bar{\rho}]^\dagger \tag{4.17}$$

These conditions are consistent with the $\star$-gauge and $\star$-supersymmetry variations (both for Majorana and chiral fermions), as in the commutative case, and can be used to check that the action (4.1) is real. The hermiticity conditions imply that the component fields V^a, $\tilde{V}^a$, ω^{ab}, ω, and $\tilde{\omega}$, and gauge parameters ε^{ab}, ε, and $\tilde{\varepsilon}$ are real fields.

4.4 Component analysis

Here we list the $\star$-gauge and supersymmetry variations of the component fields. In the supersymmetry variations we consider both Majorana and Weyl fermions.

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4.4.1 $\star$-Gauge variations

$$\delta_\epsilon V^a = \frac{1}{2}\left(\varepsilon^a{}_b \star V^b + V^b \star \varepsilon^a{}_b\right) + \frac{i}{4}\varepsilon^a{}_{bcd}\left(\tilde{V}^b \star \varepsilon^{cd} - \varepsilon^{cd} \star \tilde{V}^b\right)$$
$$+\varepsilon \star V^a - V^a \star \varepsilon - \tilde{\varepsilon} \star \tilde{V}^a - \tilde{V}^a \star \tilde{\varepsilon} \tag{4.18}$$
$$\delta_\epsilon \tilde{V}^a = \frac{1}{2}\left(\varepsilon^a{}_b \star \tilde{V}^b + \tilde{V}^b \star \varepsilon^a{}_b\right) + \frac{i}{4}\varepsilon^a{}_{bcd}\left(V^b \star \varepsilon^{cd} - \varepsilon^{cd} \star V^b\right)$$
$$+\varepsilon \star \tilde{V}^a - \tilde{V}^a \star \varepsilon - \tilde{\varepsilon} \star V^a - V^a \star \tilde{\varepsilon} \tag{4.19}$$
$$\delta_\epsilon \omega^{ab} = \frac{1}{2}\left(\varepsilon^a{}_c \star \omega^{cb} - \varepsilon^b{}_c \star \omega^{ca} + \omega^{cb} \star \varepsilon^a{}_c - \omega^{ca} \star \varepsilon^b{}_c\right)$$
$$+\frac{1}{4}\left(\varepsilon^{ab} \star \omega - \omega \star \varepsilon^{ab}\right) + \frac{i}{8}\varepsilon^{ab}{}_{cd}\left(\varepsilon^{cd} \star \tilde{\omega} - \tilde{\omega} \star \varepsilon^{cd}\right)$$
$$+\frac{1}{4}\left(\varepsilon \star \omega^{ab} - \omega^{ab} \star \varepsilon\right) + \frac{i}{8}\varepsilon^{ab}{}_{cd}\left(\tilde{\varepsilon} \star \omega^{cd} - \omega^{cd} \star \tilde{\varepsilon}\right) \tag{4.20}$$
$$\delta_\epsilon \omega = \frac{1}{8}\left(\omega^{ab} \star \varepsilon_{ab} - \varepsilon_{ab} \star \omega^{ab}\right) + \varepsilon \star \omega - \omega \star \varepsilon + \tilde{\varepsilon} \star \tilde{\omega} - \tilde{\omega} \star \tilde{\varepsilon} \tag{4.21}$$
$$\delta_\epsilon \tilde{\omega} = \frac{i}{16}\varepsilon_{abcd}\left(\omega^{ab} \star \varepsilon^{cd} - \varepsilon^{cd} \star \omega^{ab}\right) + \varepsilon \star \tilde{\omega} - \tilde{\omega} \star \varepsilon + \tilde{\varepsilon} \star \omega - \omega \star \tilde{\varepsilon} \tag{4.22}$$

4.4.2 Supersymmetry variations: Majorana fermions

$$\delta_\epsilon V^a = \frac{i}{2} Tr[(\epsilon \star \bar{\psi} - \psi \star \bar{\epsilon})\gamma^a] \tag{4.23}$$
$$\delta_\epsilon \tilde{V}^a = \frac{i}{2} Tr[(\epsilon \star \bar{\psi} - \psi \star \bar{\epsilon})\gamma^a \gamma_5] \tag{4.24}$$
$$\delta_\epsilon \psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon - (i\omega + \tilde{\omega}\gamma_5)\epsilon \tag{4.25}$$

4.4.3 Supersymmetry variations: Weyl fermions

$$\delta_\epsilon V^a = \delta_\epsilon \tilde{V}^a = \frac{i}{2} Tr[(\epsilon \star \bar{\psi} - \psi \star \bar{\epsilon})\gamma^a] \tag{4.26}$$
$$\delta_\epsilon \psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon - (i\omega + \tilde{\omega})\epsilon \tag{4.27}$$

4.4.4 Charge conjugation conditions

The charge conjugation relations (4.12) imply for the component fields:

$$V^a_\theta = V^a_{-\theta}, \qquad \omega^{ab}_\theta = \omega^{ab}_{-\theta} \tag{4.28}$$
$$\tilde{V}^a_\theta = -\tilde{V}^a_{-\theta}, \qquad \omega_\theta = -\omega_{-\theta}, \qquad \tilde{\omega}_\theta = -\tilde{\omega}_{-\theta}, \tag{4.29}$$

and for the gauge parameters:

$$\varepsilon^{ab}_\theta = \varepsilon^{ab}_{-\theta} \tag{4.30}$$
$$\varepsilon_\theta = -\varepsilon_{-\theta}, \quad \tilde{\varepsilon}_\theta = -\tilde{\varepsilon}_{-\theta} \tag{4.31}$$

4.5 Field equations and Bianchi identities

Using the cyclicity of the integral and of the Tr in the action (4.1), the variation on V, Ω and ψ yield respectively the Einstein equation, the torsion equation and the gravitino equation in index-free form:

$$Tr[\Gamma_{a,a5}(-iV \wedge_\star R - iR \wedge_\star V + 2(\rho \wedge_\star \bar{\psi} + \psi \wedge_\star \bar{\rho})] = 0 \tag{4.32}$$

$$Tr[\Gamma_{ab,1,5}(iT \wedge_\star V - iV \wedge_\star T + 2\psi \wedge_\star \bar{\psi} \wedge V - 2V \wedge_\star \psi \wedge_\star \bar{\psi})] = 0 \tag{4.33}$$

$$V \wedge_\star D\psi - \frac{1}{2} T \wedge_\star \psi = 0 \tag{4.34}$$

where $\Gamma_{ab,1,5}$ indicates γ_{ab}, 1 and γ_5 (thus there are three distinct equations) and likewise for $\Gamma_{a,a5}$ (two equations corresponding to γ_a and $\gamma_a\gamma_5$). The torsion $T = T^a\gamma_a + \tilde{T}^a\gamma_a\gamma_5$ is defined as:

$$T \equiv dV - \Omega \wedge_\star V - V \wedge_\star \Omega \tag{4.35}$$

The torsion equation can be written as:

$$[iT \wedge_\star V - iV \wedge_\star T + 2\psi \wedge_\star \bar{\psi} \wedge_\star V - 2V \wedge_\star \psi \wedge_\star \bar{\psi}, \gamma_5] = 0 \tag{4.36}$$

since the anticommutator with γ_5 selects the γ_{ab}, 1 and γ_5 components. This equation can be solved for the torsion:

$$T = i[\psi \wedge_\star \bar{\psi}, \gamma_5]\gamma_5 = i\psi \wedge_\star \bar{\psi} - i\gamma_5\psi \wedge_\star \bar{\psi}\gamma_5 \tag{4.37}$$

For chiral gravitini:

$$T = 2i\psi \wedge_\star \bar{\psi} \tag{4.38}$$

The Bianchi identities for the curvatures and the torsion are obtained from the commutative ones simply by replacing exterior products by $\star$-exterior products.

4.6 Commutative limit

The nonsupersymmetric NC theory with NC Majorana gravitino, and charge conjugation conditions (4.12), reduces in the $\theta \to 0$ limit to the usual $D = 4$, $N = 1$ supergravity. Indeed the charge conjugation conditions on V and Ω imply that the component fields $\tilde{V}^a$, ω, and $\tilde{\omega}$ all vanish in the limit $\theta \to 0$ (see the second line of (4.29)), and only the classical spin connection ω^{ab}, vierbein V^a and Majorana fermion ψ survive. Similarly the gauge parameters ε, and $\tilde{\varepsilon}$ vanish in the commutative limit.

In the chiral case, the extra vielbein $\tilde{V}^a$ cannot vanish in the commutative limit, since its supersymmetry variation is equal to that of V^a. Then one obtains a commutative limit that is a (locally) supersymmetric version of gravity with a complex vielbein studied by Chamseddine, or a bigravity-like theory (in our case a super-bigravity theory). For a discussion on chiral supergravity see for ex. [12]. A detailed study of this commutative limit will not be carried out in the present paper.

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4.7 The noncommutative supergravity action in terms of chiral fields

In the case of chiral fermions, it may be useful to reexpress the action in terms of chiral bosonic and fermionic fields. Chiral bosonic fields can be defined in exactly the same way as chiral fermionic fields, since V and Ω take values in the spinor representation (they are Clifford algebra valued fields). Thus we'll denote by $V_\pm$ and $\Omega_\pm$ the projections

$$V_\pm = \frac{1}{2}(1 \pm \gamma_5)V, \quad \Omega_\pm = \frac{1}{2}(1 \pm \gamma_5)\Omega \tag{4.39}$$

Note that the spin connection ω^{ab} contained in $\Omega_\pm$ is then (anti)self-dual.

The action (4.1) takes the form:

$$S = \int Tr[iR_+ \wedge_\star V_+ \wedge_\star V_- - iR_- \wedge_\star V_- \wedge_\star V_+ + 2(\rho \wedge_\star \bar{\psi} + \psi \wedge_\star \bar{\rho}) \wedge_\star V_-] \tag{4.40}$$

with

$$R_\pm = d\Omega_\pm - \Omega_\pm \wedge_\star \Omega_\pm \tag{4.41}$$

The transformation rules and the field equations can all be rewritten in terms of the chiral fields. For example under supersymmetry the "chiral vielbein" $V_\pm$ transform as:

$$\delta_\epsilon V_+ = 2i(\epsilon \star \bar{\psi} - \psi \star \bar{\epsilon}), \quad \delta_\epsilon V_- = 0 \tag{4.42}$$

Similarly the torsion equation becomes:

$$T_+ = 2i\psi \wedge_\star \bar{\psi}, \quad T_- = 0 \tag{4.43}$$

5 Conclusions

The index-free notation, based on Clifford algebra expansion of the bosonic fields (see for ex. ref.s [2, 12]), allows to study invariances with simple algebraic manipulations. This framework is ideally suited to study noncommutative generalizations of field theories containing gravity, cf. ref.s [2], where a complex noncommutative gravity was proposed. In ref. [8] we showed that a NC gravity could be constructed, with a commutative limit coinciding with the usual Einstein-Cartan theory. We proved that a NC charge conjugation condition on the vierbein and on the spin connection yields a real vierbein in the commutative limit. The theory was also coupled to (Majorana) fermion zero-forms (spin 1/2).

In this paper we have constructed noncommutative supergravities in $D = 3$ and $D = 4$. The commutative limit of the $D = 3$ locally supersymmetric theory coincides with pure supergravity (without cosmological term) in $D = 3$. The $D = 4$ model is less satisfactory: if we use the NC Majorana condition for the gravitino, the action is not $\star$-supersymmetric. However in this case we can impose charge conjugation conditions on the vierbein and spin connection, so that the commutative limit of the theory reproduces usual $D = 4$, $N = 1$ supergravity.

We recover $\star$-local supersymmetry of the action when the gravitino is chiral. In this case we cannot impose the charge conjugation condition on the vierbein (because then

$\star$-supersymmetry requires the NC Majorana condition on the gravitino), and therefore the commutative limit does not involve only one real vierbein, but reduces to a chiral $D = 4$, $N = 1$ supergravity with a complex vierbein.

Note that the $\star$-products deformations considered in this paper are associated to a very general triangular Drinfeld twist $\mathcal{F}$, a particular case being the Groenewold-Moyal $\star$-product. In our general framework one could consider promoting the twist $\mathcal{F}$ itself to a dynamical field, see [13] for an example in the flat case.

A Twist differential geometry

The noncommutative deformation of the gravity theories we constructed relies on the existence (in the deformation quantization context, see for ex [14]) of an associative $\star$-product between functions and more generally an associative $\wedge_\star$ exterior product between forms that satisfies the following properties:

- Compatibility with the undeformed exterior differential:

$$d(\tau \wedge_\star \tau') = d(\tau) \wedge_\star \tau' + (-1)^{deg(\tau)} \tau \wedge_\star d\tau' \tag{A.1}$$

- Compatibility with the undeformed integral (graded cyclicity property):

$$\int \tau \wedge_\star \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{A.2}$$

  with $deg(\tau)+deg(\tau')$ =D=dimension of the spacetime manifold, and where here τ and τ' have compact support (otherwise stated we require (A.2) to hold up to boundary terms).

- Compatibility with the undeformed complex conjugation:

$$(\tau \wedge_\star \tau')^* = (-1)^{deg(\tau)deg(\tau')} \tau'^* \wedge_\star \tau^* \tag{A.3}$$

We describe here a (quite wide) class of twists whose $\star$-products have all these properties. In this way we have constructed a wide class of noncommutative deformations of gravity theories. Of course as a particular case we have the Groenewold-Moyal $\star$-product

$$f \star g = \mu \left\{ e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g \right\}, \tag{A.4}$$

where the map μ is the usual pointwise multiplication: $\mu(f \otimes g) = fg$, and $\theta^{\rho\sigma}$ is a constant antisymmetric matrix.

Twist. Let Ξ be the linear space of smooth vector fields on a smooth manifold M, and $U\Xi$ its universal enveloping algebra. A twist $\mathcal{F} \in U\Xi \otimes U\Xi$ defines the associative twisted product

$$f \star g = \mu \left\{ \mathcal{F}^{-1} f \otimes g \right\} \tag{A.5}$$

where the map μ is the usual pointwise multiplication: $\mu(f\otimes g) = fg$. The product associativity relies on the defining properties of the twist [3, 14, 15]. Using the standard notation

$$\mathcal{F} \equiv \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha, \quad \mathcal{F}^{-1} \equiv \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha \tag{A.6}$$

(sum over α understood) where $\mathrm{f}^\alpha, \mathrm{f}_\alpha, \bar{\mathrm{f}}^\alpha, \bar{\mathrm{f}}_\alpha$ are elements of $U\Xi$, the $\star$-product is expressed in terms of ordinary products as:

$$f \star g = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(g) \tag{A.7}$$

Many explicit examples of twist are provided by the so-called abelian twists:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a\otimes X_b} \tag{A.8}$$

where $\{X_a\}$ is a set of mutually commuting vector fields globally defined on the manifold,[5] and θ^{ab} is a constant antisymmetric matrix. The corresponding $\star$-product is in general position dependent because the vector fields X_a are in general x-dependent. In the special case that there exists a global coordinate system on the manifold we can consider the vector fields $X_a = \frac{\partial}{\partial x^a}$. In this instance we have the Moyal twist, cf. (A.4):

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} \tag{A.9}$$

Deformed exterior product. The deformed exterior product between forms is defined as

$$\tau \wedge_\star \tau' \equiv \bar{\mathrm{f}}^\alpha(\tau) \wedge \bar{\mathrm{f}}_\alpha(\tau') \tag{A.10}$$

where $\bar{\mathrm{f}}^\alpha$ and $\bar{\mathrm{f}}_\alpha$ act on forms via the Lie derivatives $\mathcal{L}_{\bar{\mathrm{f}}^\alpha}$, $\mathcal{L}_{\bar{\mathrm{f}}_\alpha}$ (Lie derivatives along products $uv\cdots$ of elements of Ξ are defined simply by $\mathcal{L}_{uv\cdots} \equiv \mathcal{L}_u\mathcal{L}_v\cdots$). This product is associative, and in particular satisfies:

$$\tau \wedge_\star h \star \tau' = \tau \star h \wedge_\star \tau', \quad h\star(\tau\wedge_\star\tau') = (h\star\tau)\wedge_\star\tau', \quad (\tau\wedge_\star\tau')\star h = \tau\wedge_\star(\tau'\star h) \tag{A.11}$$

where h is a 0-form, i.e. a function belonging to $Fun(M)$, the $\star$-product between functions and one-forms being just a particular case of (A.10):

$$h \star \tau = \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(\tau), \quad \tau \star h = \bar{\mathrm{f}}^\alpha(\tau)\bar{\mathrm{f}}_\alpha(h) \tag{A.12}$$

Exterior derivative. The exterior derivative satisfies the usual (graded) Leibniz rule, since it commutes with the Lie derivative:

$$d(f\star g) = df \star g + f \star dg \tag{A.13}$$

$$d(\tau\wedge_\star\tau') = d\tau\wedge_\star\tau' + (-1)^{deg(\tau)}\,\tau\wedge_\star d\tau' \tag{A.14}$$

[5] We actually need only the twist $\mathcal{F}$ to be globally defined, not necessarily the single vector fields X_a. An explicit example of this latter kind is given by the twist (A.8), that in an open neighbourhood with coordinates t,x,y,z is defined by the commuting vector fields $X_1 = f(x,z)\frac{\partial}{\partial x}$, $X_2 = h(y,z)\frac{\partial}{\partial y}$, where $f(x,z)$ is a function of only the x and z variables and has compact support, and similarly $h(y,z)$. This twist is globally defined on the whole manifold by requiring it to be the identity $1\otimes 1$ outside the $\{x^a\}$ coordinate neighbourhood. The corresponding $\star$-product, defined on the whole spacetime manifold, is noncommutative only inside this neighbourhood.

Integration: graded cyclicity. If we consider an abelian twist (A.8) given by globally defined commuting vector fields X_a, then the usual integral is cyclic under the $\star$-exterior products of forms, i.e., up to boundary terms,

$$\int \tau \wedge_\star \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{A.15}$$

with $deg(\tau) + deg(\tau')$ =D=dimension of the spacetime manifold. In fact we have

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge \tau = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{A.16}$$

For example at first order in θ,

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' - \frac{i}{2}\theta^{ab} \int \mathcal{L}_{X_a}(\tau \wedge \mathcal{L}_{X_b}\tau') = \int \tau \wedge \tau' - \frac{i}{2}\theta^{ab} \int d i_{X_a}(\tau \wedge \mathcal{L}_{X_b}\tau') \tag{A.17}$$

where we used the Cartan formula $\mathcal{L}_{X_a} = d i_{X_a} + i_{X_a} d$.

More generally if the twist $\mathcal{F}$ satisfies the condition $S(\bar{\mathrm{f}}^\alpha)\bar{\mathrm{f}}_\alpha = 1$, where the antipode S is defined on vector fields as $S(v) = -v$ and is extended to the whole universal enveloping algebra $U\Xi$ linearly and antimultiplicatively, $S(uv) = S(v)S(u)$, then a similar argument proves the graded cyclicity of the integral.[6]

Complex conjugation. If we choose real fields X_a in the definition of the twist (A.8), it is immediate to verify that:

$$(f \star g)^* = g^* \star f^* \tag{A.18}$$

$$(\tau \wedge_\star \tau')^* = (-1)^{deg(\tau)deg(\tau')} \tau'^* \wedge_\star \tau^* \tag{A.19}$$

since sending i into $-i$ in the twist (A.9) amounts to send θ^{ab} into $-\theta^{ab} = \theta^{ba}$, i.e. to exchange the order of the factors in the $\star$-product.

More in general we can consider twists $\mathcal{F}$ that satisfy the reality condition (cf. section 8 in [3]) $\bar{\mathrm{f}}^{\alpha *} \otimes \bar{\mathrm{f}}_\alpha{}^* = S(\bar{\mathrm{f}}_\alpha) \otimes S(\bar{\mathrm{f}}^\alpha)$. The $\star$-products associated to these twists satisfy properties (A.18), (A.19).

B Gamma matrices in $D = 3$

We summarize in this appendix our gamma matrix conventions in $D = 3$.

$$\gamma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.1}$$

$$\eta_{ab} = (-1, 1, 1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab} = -2\varepsilon_{abc}\gamma^c, \tag{B.2}$$

$$\varepsilon_{012} = -\varepsilon^{012} = 1, \tag{B.3}$$

$$\gamma_a^\dagger = \gamma_0 \gamma_a \gamma_0 \tag{B.4}$$

[6]Proof: using Sweedler's coproduct notation we have (cf. footnote 3 in [8] that goes into details)

$$\begin{aligned} \tau \wedge_\star \tau' &= \bar{\mathrm{f}}^\alpha(\tau) \wedge \bar{\mathrm{f}}_\alpha(\tau') = \bar{\mathrm{f}}_1^\alpha(\tau \wedge S(\bar{\mathrm{f}}_2^\alpha)\bar{\mathrm{f}}_\alpha(\tau')) = \tau \wedge S(\bar{\mathrm{f}}^\alpha)\bar{\mathrm{f}}_\alpha(\tau') + \bar{\mathrm{f}}^{\alpha\prime}{}_1(\tau \wedge S(\bar{\mathrm{f}}^{\alpha\prime}{}_2)\bar{\mathrm{f}}_\alpha(\tau')) \\ &= \tau \wedge \tau' + \text{total derivative} \end{aligned}$$

where $\Delta \bar{\mathrm{f}}^\alpha \equiv \bar{\mathrm{f}}_1^\alpha \otimes \bar{\mathrm{f}}_2^\alpha \equiv 1 \otimes \bar{\mathrm{f}}^\alpha + \bar{\mathrm{f}}^{\alpha\prime}{}_1 \otimes \bar{\mathrm{f}}^{\alpha\prime}{}_2$, and in the last equality we observe that each $\bar{\mathrm{f}}^{\alpha\prime}{}_1$ contains at least one vector field. Thus use of Cartan's formula implies that the second addend is a total derivative.

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B.1 Useful identities

$$\begin{aligned}
\gamma_a\gamma_b &= \gamma_{ab}+\eta_{ab} = -\varepsilon_{abc}\gamma^c+\eta_{ab} && \text{(B.5)}\\
\gamma_{ab}\gamma_c &= \eta_{bc}\gamma_a-\eta_{ac}\gamma_b-\varepsilon_{abc} && \text{(B.6)}\\
\gamma_c\gamma_{ab} &= \eta_{ac}\gamma_b-\eta_{bc}\gamma_a-\varepsilon_{abc} && \text{(B.7)}\\
\gamma_a\gamma_b\gamma_c &= \eta_{ab}\gamma_c+\eta_{bc}\gamma_a-\eta_{ac}\gamma_b-\varepsilon_{abc} && \text{(B.8)}\\
\gamma^{ab}\gamma_{cd} &= -4\delta^{[a}_{[c}\gamma^{b]}{}_{d]}-2\delta^{ab}_{cd} && \text{(B.9)}
\end{aligned}$$

where $\delta^{ab}_{cd}=\frac{1}{2}(\delta^a_c\delta^b_d-\delta^a_d\delta^b_c)$, and index antisymmetrizations in square brackets have weight 1.

C Gamma matrices in $D = 4$

We summarize in this appendix our gamma matrix conventions in $D = 4$.

$$\begin{aligned}
\eta_{ab} &= (1,-1,-1,-1), \quad \{\gamma_a,\gamma_b\}=2\eta_{ab}, \quad [\gamma_a,\gamma_b]=2\gamma_{ab}, && \text{(C.1)}\\
\gamma_5 &\equiv i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5\gamma_5=1, \quad \varepsilon_{0123}=-\varepsilon^{0123}=1, && \text{(C.2)}\\
\gamma_a^\dagger &= \gamma_0\gamma_a\gamma_0, \quad \gamma_5^\dagger=\gamma_5 && \text{(C.3)}\\
\gamma_a^T &= -C\gamma_aC^{-1}, \quad \gamma_5^T=C\gamma_5C^{-1}, \quad C^2=-1, \quad C^T=-C && \text{(C.4)}
\end{aligned}$$

C.1 Useful identities

$$\begin{aligned}
\gamma_a\gamma_b &= \gamma_{ab}+\eta_{ab} && \text{(C.5)}\\
\gamma_{ab}\gamma_5 &= \frac{i}{2}\epsilon_{abcd}\gamma^{cd} && \text{(C.6)}\\
\gamma_{ab}\gamma_c &= \eta_{bc}\gamma_a-\eta_{ac}\gamma_b-i\varepsilon_{abcd}\gamma_5\gamma^d && \text{(C.7)}\\
\gamma_c\gamma_{ab} &= \eta_{ac}\gamma_b-\eta_{bc}\gamma_a-i\varepsilon_{abcd}\gamma_5\gamma^d && \text{(C.8)}\\
\gamma_a\gamma_b\gamma_c &= \eta_{ab}\gamma_c+\eta_{bc}\gamma_a-\eta_{ac}\gamma_b-i\varepsilon_{abcd}\gamma_5\gamma^d && \text{(C.9)}\\
\gamma^{ab}\gamma_{cd} &= -i\varepsilon^{ab}{}_{cd}\gamma_5-4\delta^{[a}_{[c}\gamma^{b]}{}_{d]}-2\delta^{ab}_{cd} && \text{(C.10)}
\end{aligned}$$

C.2 Charge conjugation and Majorana condition

$$\begin{aligned}
\text{Dirac conjugate} \quad \bar\psi &\equiv \psi^\dagger\gamma_0 && \text{(C.11)}\\
\text{Charge conjugate spinor} \quad \psi^c &= C(\bar\psi)^T && \text{(C.12)}\\
\text{Majorana spinor} \quad \psi^c &= \psi \quad \Rightarrow \bar\psi=\psi^TC && \text{(C.13)}
\end{aligned}$$

C.3 Fierz identities for two spinor one-forms

$$\psi\wedge\bar\chi = \frac{1}{4}\left[(\bar\chi\wedge\psi)\,1+(\bar\chi\gamma_5\wedge\psi)\,\gamma_5+(\bar\chi\gamma^a\wedge\psi)\,\gamma_a+(\bar\chi\gamma^a\gamma_5\wedge\psi)\,\gamma_a\gamma_5-\frac{1}{2}\left(\bar\chi\gamma^{ab}\wedge\psi\right)\gamma_{ab}\right] \tag{C.14}$$

Noncommutative Fierz identities.

$$\begin{aligned}
\psi\wedge_\star\bar\chi = \frac{1}{4}\Big[&Tr\,(\psi\wedge_\star\bar\chi)\,1+Tr\,(\psi\gamma_5\wedge_\star\bar\chi)\,\gamma_5+Tr\,(\psi\gamma^a\wedge_\star\bar\chi)\,\gamma_a+\\
&Tr\,(\psi\gamma^a\gamma_5\wedge_\star\bar\chi)\,\gamma_a\gamma_5-\frac{1}{2}Tr\left(\psi\gamma^{ab}\wedge_\star\bar\chi\right)\gamma_{ab}\Big] && \text{(C.15)}
\end{aligned}$$

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